The specified function $f(\tau)$ is characteristic of salt baths in which tools are heated for heat treatment. Figure 1 presents the calculated dependences of the temperature of the source ensuring the regularity of heating the surface $T_s = f(\tau)$ for specimens with radius 0.005, 0.01, and 0.02 m.

Figure 2 shows the calculated and the experimental dependences of the temperature of the source and of the surface for specimens with radius 0.005 m.

The experimental installation for radiative heating of objects and the experimental method were presented by us in [3]. It can be seen from Fig. 2 that the experiment is in good agreement with the calculation.

The obtained result is of great practical importance because controlled radiative heating can be instrumental in attaining specified regimes of heat treatment so that ecologically harmful heat sources can be replaced in production.

NOTATION

$T = T(\bar{r}, \tau)$ is temperature at the point with the radius vector $\bar{r}$ at the instant τ; T_{so} and T_s) temperature of the source and of the surface, respectively; c_p) specific heat; ρ) density; λ) thermal conductivity; ε) reduced degree of blackness; σ_0) Stefan–Boltzmann constant.

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NONSTEADY THERMOELASTIC DEFORMATION OF COOLED LASER MIRRORS

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UDC 539.377:621.375.826

The article investigates the temperature fields and distortions of the optical surface of laser mirrors of different materials subjected to different kinds of thermal loading.

The designing, production, and testing of high-energy laser mirrors is very laborious and expensive. That is why various engineering–physical models are worked out for them, making it possible to determine theoretically distortions of the shape of the optical surface of reflectors in dependence on their geometric dimensions, properties of the materials, the structure of the cooling system, the flow rate of the heat carrier, the profile of the luminous load, etc. [1-3]. Most fully developed are methods of calculating round mirrors subjected to axisymmetric beam load, based on the examination of steady temperature fields. It was shown in [3] that for solid disk mirrors cooled from the rear surface there exists an exact solution of the spatial axisymmetric problem of thermoelasticity. Nonsteady deformation of uniformly illuminated reflectors was dealt with in [4] in the approximation of the theory of thermoelasticity of thin plates.

In the present work we explain the method of calculating three-dimensional nonsteady distributions of temperature and of thermoelastic displacements in cooled mirrors that have the shape of parallelepipeds (Fig. 1). The method takes into account the special traits of spatial distribution of the thermal load, the anisotropy of the thermal conductivities,

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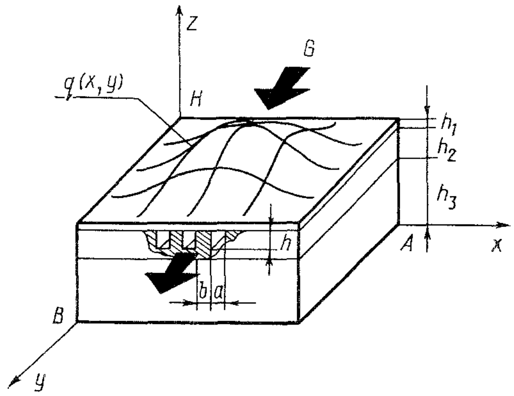

Fig. 1. Calculation diagram of the cooled mirror: h_1) base, h_2) cooled layer, h_3) reflecting layer.

heating of the heat carrier while it flows through the cooling system. We present the nontrivial results of calculation of thermal displacements of the reflecting surface of rectangular mirrors with channel (silt) cooling system.

1. Method of Calculating Temperature Fields. The mirror in the form of a parallelepiped with sides A, B, and H is loaded in the plane z = H by a thermal flux with density $\tilde{q}(x, y)$, W/m^2 forming in the reflecting layer a temperature gradient in accordance with Fourier's law: $q(x, y) = -\lambda_z \partial T/\partial z$ for z = H. The other outer surfaces (sides) are heat-insulated, i.e., the temperature gradients normal to these surfaces are equal to zero. The heat carrier is pumped through the cooling system in the direction of the y-axis at the rate G, kg/sec. The cause of deformation is the thermal expansion of the material. It is known [5] that here we may neglect the influence of deformation on the temperature field and solve the problem of heat conduction and of elasticity independently (i.e., within the framework of the unconnected theory of thermoelasticity). The equation of nonsteady heat conduction describing the temperature distribution in each layer was used in the approximation of an anisotropic porous medium [2]:

$$\rho c \frac{\partial T}{\partial \tau} = \lambda_x \frac{\partial^2 T}{\partial x^2} + \lambda_y \frac{\partial^2 T}{\partial y^2} + \lambda_z \frac{\partial^2 T}{\partial z^2} - \alpha_v (T - T_\ell), \tag{1}$$

where $T(x, y, z, \tau)$ is the temperature of the mirror; $T_\ell(x, y, \tau)$ is the temperature of the liquid (the heat carrier) in the cooling system. The term $\alpha_v(T - T_\ell)$ takes into account the volumetric heat sinks in the homogenized cooled layer whose ribs (carcass) are cooled in accordance with the Newton–Richman law [2, 4, 6]. In the reflecting layer and in the base of the mirror, where there are no volumetric heat sinks, the volumetric heat transfer coefficient $\alpha_v = 0$. On the interfaces between layers, like everywhere inside the mirror, the temperature and the heat flux density are continuous functions. The initial state of the mirror is usually isothermal at some temperature T_0, which is not mandatorily equal to the temperature of the liquid $T_{\ell 0}$ at the inlet to the cooled layer.

This is briefly the mathematical statement of the thermal problem when the temperature distribution of the liquid $T_\ell(x, y, \tau)$ is known. Since T_ℓ is not known beforehand, it can be determined from the point solution of the equation of heat conduction (1) and the equation of the material and thermal balance for the cooled layer

$$h \rho_\ell c_\ell v \frac{\partial T_\ell}{\partial y} = \int_h \alpha_v (T - T_\ell)\, dz, \tag{2}$$

and at the inlet to the cooled layer (y = 0) we have $T_\ell(x, 0, \tau) = T_{\ell 0}$. To reduce the number of arithmetic operations in the calculation of the temperatures we used a semianalytical method: the solution of the three-dimensional equation of heat conduction (1) was sought in the form of a double Fourier series:

$$T(x, y, z, \tau) = T_0 + \sum_{m,n=0}^{\infty} T_{mn}(z, \tau) \cos\frac{\pi m x}{A} \cos\frac{\pi n y}{B}, \tag{3}$$

where $T_{mn}(z, \tau)$ are unknown functions. Substituting expression (3) into (1), we obtain the following equations, initial and boundary conditions for determining the functions T_{mn}:

$$\rho c \frac{\partial T_{mn}}{\partial \tau} = \lambda_z \frac{\partial^2 T_{mn}}{\partial z^2} - \left(\lambda_x \frac{\pi^2 m^2}{A^2} + \lambda_y \frac{\pi^2 n^2}{B^2} + \alpha_V\right) T_{mn} + \alpha_V T_{\ell mn}; \tag{4}$$

$$T_{mn}(z, 0) = 0; \quad \left.\frac{\partial T_{mn}}{\partial z}\right|_{z=H} = \frac{q_{mn}}{\lambda_z}; \quad \left.\frac{\partial T_{mn}}{\partial z}\right|_{z=0} = 0, \tag{5}$$

where q_{mn} and $T_{\ell mn}$ are the coefficients of expansion of the functions $q(x, y)$ and $T_\ell(x, y)$ into a double Fourier series in the form of (3). On the interface of the cooled layer with the reflecting plate and with the base, conditions of continuity of temperature and of heat flux were specified.

Equations (4) are unidimensional parabolic nonsteady-state equations of heat conduction which are solved by the finite difference method with the use of the method of matching. The number of equations in (4) to be solved corresponds to the number of terms in expression (3). The summation limits of (3) depend on the degree of detailing of the function $q(x, y)$, they are limited from above by the possibilities of the computer, and in our case they are equal to 20. To find the solutions of Eq. (4), we have to know the temperature of the heat carrier in the cooling system $T_\ell(x, y)$. This can be found most simply and economically by integrating the equation of material and thermal balance (2) after each step with respect to time, i.e., by taking into account the amount of heat obtained by the heat carrier from the carcass of the cooled layer at the preceding time step:

$$T_\ell(x, y, \tau_{i+1}) = T_{\ell 0} + \int_0^y \frac{\alpha_V [\overline{T}(x, y', \tau_i) - T_\ell(x, y', \tau_i)]}{\rho_\ell c_\ell v} dy'. \tag{6}$$

Here, $\overline{T}(x, y, \tau_i)$ is the temperature of the carcass (ribs) of the cooled layer averaged over the z-coordinate; i is the number of the time step; y' is the integration variable ($0 \le y' \le y$).

The thermophysical parameters of the homogenized cooled layer (α_V, λ_x, λ_y, λ_z) were determined by a method from [2, 6] coordinated with various experimental data. For instance, for a layer with slit (channel) structure, like in Fig. 1, we have approximately $\lambda_\ell = 0$; $\lambda_y = \lambda_z = \lambda(1 - \Pi)$, where λ is the thermal conductivity of the material of the ribs; $\Pi = a/(a + b)$ is the porosity of the cooling system; a is the thickness of the channel; b is the thickness of the rib. The volumetric heat transfer coefficient is correlated with the heat transfer coefficient on the surface of the ribs α, $W/(m^2 \cdot K)$, calculated by the traditional formulas for flat channels (with rectangular cross section), by the approximate expression $\alpha_V = \alpha 2(h + a)/h(a + b) = (\alpha/h)2\Pi(1 + h/a)$. Stricter and more complex methods are presented in [6] and in the Appendix of the present article.

<u>2. Method of Calculating Thermal Displacements.</u> In the solution of the problem of thermoelasticity the boundary conditions were provided by the assumption that the normal stress components on all the outer surfaces of the mirror are equal to zero, which corresponds to the frequently encountered case when the fastening is much less rigid than the reflector itself. To find the displacement fields in the present work, we use the finite element method based on the minimization of the potential strain energy which was described in detail in the monographs [7, 8]. The finite elements had the shape of a tetrahedron with four nodal points. The temperature at the nodal points at different isntants was calculated beforehand by the method explained above.

The sought components of the displacement vector were approximated by linear functions within the limits of each finite element. The finite elements were grouped into elementary parallelepipeds of five elements each. The network for calculating the displacements, formed by the nodes of the finite elements, had the following parameters: 21 points on the x- and y-axes, 7 points on the z-axis. In solving the system of linear algebraic equations obtained in the process of realization of the finite element method we gave preference to the iteration method of successive superrelaxations because it requires much less capacity of the internal computer memory. To reduce the number of iterations, we found beforehand the approximate displacements using the relations of the theory of thermoelasticity of thin plates. Good convergence of the iteration process was attained after 200-400 iterations with the parameter of superrelaxation equal to 1.7. The displacements were measured from a plane determined

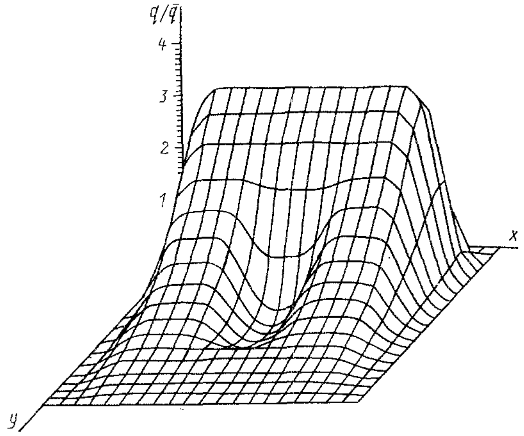

Fig. 2

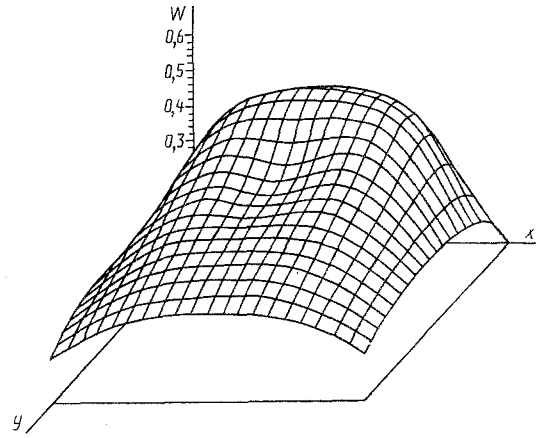

Fig. 3

Fig. 2. Spatial distribution of the radiation flux density onto the reflecting surface of the mirror.

Fig. 3. The z-component of displacements of the reflecting surface of the mirror (μm) of molybdenum for $\bar{q}$ = 100 W/cm^2; a = b = 1 mm; $\alpha_V = 4 \cdot 10^7$ W/(m^3·K). For the profile of the thermal load see Fig. 2. W, μm.

by three points of the rear surface of the mirror. The disposition of these points was chosen such that it corresponded to the disposition of the real fastenings of the reflector.

The approaches described above were realized in the form of programs in FORTRAN ensuring dialogue regime of operation and display of the results of the calculations in graphic form. The computer code of the program requires about 600 Kbytes of internal memory, and another approximately 400 Kbytes are needed for storing the variables. The correctness of the algorithms and of the programs was verified by comparison with the analytical solutions of the equations of heat conduction and of thermoelasticity in the case of unidimensional temperature distributions. The comparisons showed that the temperature fields coincide almost completely, and the displacement fields showed good agreement with an accuracy of 5%.

3. Influence of the Nonuniformity of Luminous Loading and Unsteadiness of the Temperature Fields on the Deformation of the Mirror. Below we present some results of calculations of a mirror operating under conditions that are typical of a shaping optical system of a gas-dynamical or electric-discharge CO_2-laser with unstable confocal resonator and rectangular mirrors. A typical distribution of power density of the radiation on the reflecting surface of the mirror is presented in Fig. 2. Calculations show that the profile of the temperature distribution of the reflecting surface almost coincides with the load profile; this makes it possible in principle to find the distribution of the thermal load by measuring the temperature of the working surface of the cooled reflector. Figure 3 presents the z-component of displacements of the optical surface of the reflector. It can be seen that the total flexure of the mirror as well as expansion of the subsurface layers contribute to distortion, and the two contributions are commensurable with each other. The amplitude of displacements has its maximum with Fo ≈ 0.1 (Fig. 4), then it decreases somewhat on account of more uniform heating of the base of the mirror when the loading time is longer. This result is in satisfactory agreement with the analytical evaluations in [4] where the value Fo ≈ 0.1, at which the deformations are maximal, was determined for the first time. It can be seen that the maximal amplitude of distortions is 40-50% greater than the distortions with steady temperature fields. Nonuniform thermal loading causes much greater displacements than uniform loading with the same integral power. The difference may attain a factor of 2-4 at different instants for the mirrors dealt with by us.

Figure 5 shows the responses of the shape of the reflecting surface to different profiles of thermal loading for mirrors with a cooling system concentrated near the working surface, and for mirrors cooled from the rear side. The results are presented for steady-state temperature distributions. The special features of the loading profile manifest themselves more on the shape of the surface with the first alternative of cooling (with

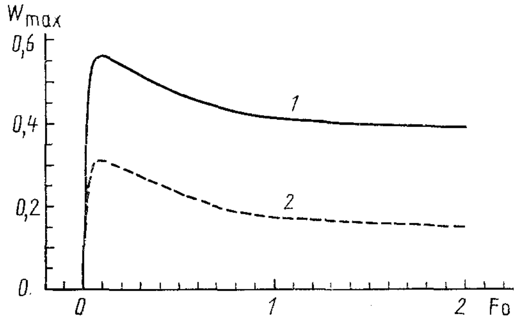

Fig. 4. Dependence of the amplitude of distortions of the optical surface (μm) on the Fourier number Fo = $\kappa\tau/H^2$: 1) nonuniform load; 2) uniform load.

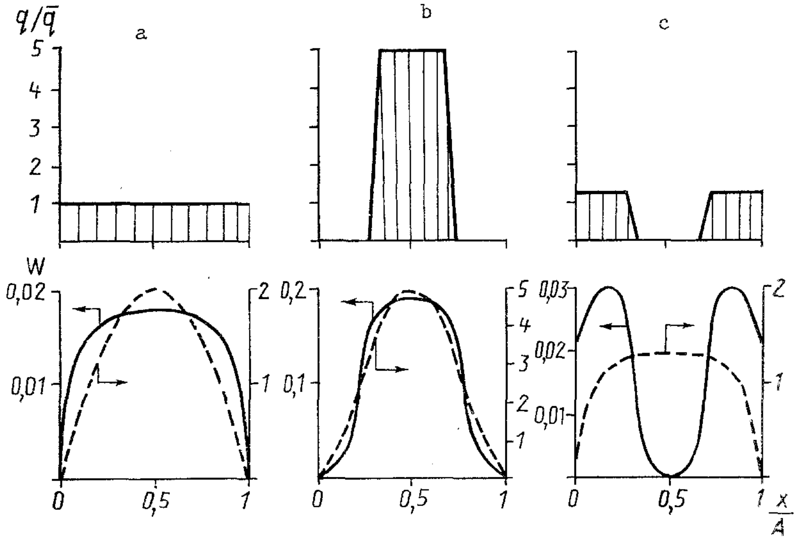

Fig. 5. Influence of the profile of thermal loading (with equal full power) and of the disposition of the cooling system on the profile of the reflecting surface of mirrors: a) uniform heating; b) heating of the central zone; c) annular heating. Solid lines below) displacements (μm) for mirrors with a cooling system near the reflecting surface (the scale on the left); dashed lines) displacements (μm) for mirrors cooled from the rear side (the scale on the right).

considerably lower level of displacements). When the center of the mirror is shaded (Fig. 5c) and cooling is effected from the rear side, there is no negative deflection of the central part of the reflecting surface; this is in full agreement with the exact solution of [3]. With cooling near the working surface such deflection occurs. Another trait is the noticeable "piling up" of the loaded edges of the mirror (see Fig. 5a, b).

4. Influence of the Choice of the Material of Structures and of the Flow Rate of Heat Carrier on the Thermal Deformation of Mirrors. When load-bearing mirrors are devised, the choice of material is particularly important. When the thermal load is moderate (less than 200-300 W/cm^2), the main criterion of serviceability of reflectors is their geometric stability. For a cooled mirror loaded by a heat flux with distribution (according to Fig. 2) with maximal density 25 W/cm^2 we calculated alternatives of materials that are traditional for load-bearing optical components: copper, beryllium, molybdenum, silicon, and Invar. The results of the calculation are presented in Table 1. When we compare the distortions with the properties of the listed materials, we come to the conclusion that the ratio β/λ of the temperature coefficient of linear expansion to the thermal conductivity, which is used for evaluating the suitability of the material of reflectors operating in a continuous regime, is ineffective where cooled mirrors are concerned. For instance, copper and molybdenum mirrors, which have similar values of β/λ and equal overall dimensions and structure of the cooled layer, differ almost three times from each other in the amplitude of displacements. The magnitude of β affects the geometric stability of cooled relflectors much more than thermal conductivity does; for mirrors of traditional design (thin reflecting plate and cooled layer, thick base) the displacements of the reflecting surface are approxiamtely proportional

TABLE 1. Maximal Displacements of the Reflecting Surface of Mirrors from Different Materials for $\bar{q}$ = 25 W/cm²

Material of mirror	Maximal displacement, μm
Copper	1,6
Beryllium	1,2
Molybdenum	0,55
Silicon	0,30
Invar	0,15

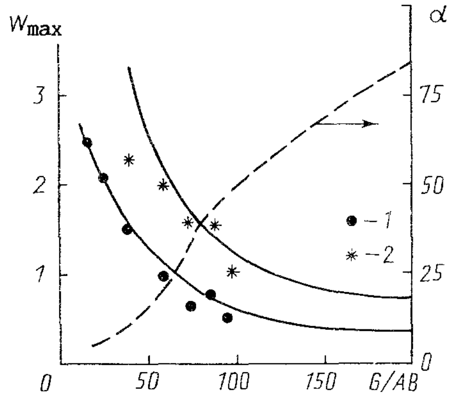

Fig. 6. Dependence of the maximal displacements (μm) and of the heat transfer coefficient (kW/(m²·K)) in the channels of the cooling system on the specific flow rate of heat carrier (kg/(m²·sec)): 1) measured displacements with q_{max} = 135 W/cm²; 2) with q_{max} = 270 W/cm²; solid lines) numerical calculation of displacements; dashed line) calculation of heat transfer.

to $\beta/\lambda^{0.3}$ (this magnitude does not have any concrete physical meaning, it is the result of generalization of numerical experiments). The weak influence of thermal conductivity is due to the fact that when the thermal conductivity of the material is lower, increased displacements due to stronger heating of the reflecting plate is largely compensated by reduced leakage of heat to the base and its correspondingly reduced deformation. In accordance with the above-said, mirrors of silicon and Invar have the best geometric stability. It should be noted that the comparatively low thermal conductivity of Invar is the cause of higher temperatures (by 30°C) of the working surface, which in this case, however, is still acceptable.

Another important aspect in the designing of mirrors and entire laser systems is the dependence of thermal displacements on the flow rate of the heat carrier in the cooling system (see Fig. 6). The figure also presents the values of the experimentally measured displacements in experiments with electron-beam heating of mirrors. The cooling system of the molybdenum mirror consists of straight rectangular channels (see Fig. 1) 0.6 mm wide, 2.5 mm deep, and with a pitch of 1.4 mm. The nature of the presented dependences shows that from some values onward, a further increase of the flow rate of the heat carrier (or heat exchange intensified by other methods) has little effect.

Conclusion. In the briefly described method of calculating the three-dimensional nonsteady distributions of temperatures and of thermoelastic displacements in cooled mirrors with the shape of parallelepipeds, a number of procedures substantially reducing computer time was used:

homogenization of the heterogeneous structure of some layers (e.g., the cooled layer);

combined analytical and numerical methods of calculating the temperature (application of a double Fourier series with respect to the x-, y-coordinates);

calculation of the distribution of the temperature of a liquid on the basis of the balance relations for the preceding time step;

application of the iteration method of superrelaxation for solving systems of algebraic equations arising in the realization of the finite element method in the problem of thermoelasticity.

The actual results obtained by this method and presented in Figs. 3-6 show that reliable calculation of thermal displacements of the reflecting surface of laser mirrors requires that factors such as nonuniformity of the beam loading, nonsteadiness of thermal processes, the structure of the cooled layer, the flow rate of the heat carrier, and the composition of the structural materials be sensitively taken into account.

The program of numerical calculation of mirrors worked out with a view to these factors makes it possible to optimize the geometric dimensions of the cooling system and the flow rate of heat carrier for each possible structural material and for various profiles of beam loading. This program is also suitable for the analysis of test results of mirrors.

Appendix. Method of Determining the Thermophysical Parameters of the Homogenized Layer.

In the directions x, y the thermal conductivities are equal to $\lambda_x = 0$, $\lambda_y = \lambda(1-\Pi)$. Thermal conductivity λ_z and volumetric heat transfer α_V can be determined from the comparison of the analytically and numerically determined temperature of the ribs for the case of steady-state heating of a mirror by a uniform heat flux. According to [2, 6] the analytical solution of the unidimensional (along the z-axis) equation of heat conduction (1) for a homogenized layer with volumetric heat sinks under uniform thermal loading has the form

$$T(z, y) = T_\ell(y) + q\frac{\mathrm{ch}(z'/\delta)}{\alpha_c \,\mathrm{sh}(h/\delta)}, \tag{A1}$$

where $\delta = \sqrt{\lambda_z/\alpha_V}$, $\alpha_c = \sqrt{\lambda_z\alpha_V} = \lambda_z/\delta$ are dimensional parameters (m; W/(m²·K)) depending on the sought λ_z and α_V (their physical meaning was explained in detail in [2]); $z' = z - (H - h - h_3)$ is the coordinate measured from the lower base of the rib (see Fig. 1); $T_\ell(y) = T_{\ell 0} + qy/\rho_\ell c_\ell vh$ is the temperature of the liquid which rises linearly with the distance y from the inlet.

From (A1) we find the temperature of the lower (ℓo) and upper (u) base of the ribs (i.e., the mean temperatures of the lower and upper sides of the cooled layer at the given distance y from the inlet):

$$T_{\ell o} = T_\ell + q\frac{1}{\alpha_c \,\mathrm{sh}(h/\delta)}; \quad T_u = T_\ell + q\frac{1}{\alpha_c \,\mathrm{th}(h/\delta)}. \tag{A2}$$

These same temperatures can be determined by numerically solving the two-dimensional equation of heat conduction for a single cell of the mirror with a rib when the heat transfer coefficient α on its lateral surface and on the walls between the ribs under the same thermal load is known. Substituting the numerically determined temperatures $T_{\ell o}$ and T_u into formula (A2), we obtain two equations with the two unknowns α_c, δ or α_V, λ_z. The values of α_V and λ_z thus calculated are best able to match the analytical and numerical calculations of the temperatures on the boundaries of the cooled layers under uniform thermal loading. These values of α_V and λ_z are used in subsequent calculations of the temperature of a mirror with the same structure of the cooled layer under nonsteady conditions with an arbitrary profile of thermal loading.

A comparison of the values of α_V and λ_z determined by the explained method and by the simplified formulas presented in the article shows that both methods yield similar results when the value of α is specified.

NOTATION

x, y, z) spatial coordinates; τ) time; q) heat flux density, W/m²; G) mass flow rate of the heat carrier, kg/sec; T) temperature; λ_x, λ_y, λ_z) thermal conductivities of the material of the mirror in the directions x, y, z, respectively, W/(m·K); ρ, c, β, $\kappa = \lambda/\rho c$) density (kg/m³), heat capacity (J/kg·K), temperature coefficient of linear expansion (1/K), and thermal diffusivity (m²/sec) of the material of the mirror, respectively; α_V) volumetric heat transfer coefficient, W/(m³·K); v) filtering rate of the heat carrier, m/sec; $\mathrm{Fo} = \kappa\tau/H^2$) Fourier number.

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OPTIMAL CONTROL OF INGOT HEATING IN FLOW-THROUGH FURNACES

V. B. Kovalevskii, F. M. Babushkin, and O. Yu. Sedyako UDC 621.1.018

The algorithm developed here allows the ingot-heating conditions in a furnace corresponding to the least skin formation to be determined. The influence of the furnace atmosphere on skin growth in heating is investigated.

Economic efficiency – in particular, the choice of optimal technological conditions with respect to specified criteria – is one of the most important questions in studying heating-furnace operation. In considering the optimal control of ingot heating, effective algorithms have been developed for linear problems [1]. For nonlinear problems, it is difficult to obtain a control algorithm. However, the introduction of nonlinear components in the mathematical description of the process permits the construction of a more adequate mathematical model. Consider the nonlinear problem of skin minimization for bodies with a small internal resistance

$$\frac{dT}{dt} = k'\left(\alpha\left(T_c(t) - T(t)\right) + \sigma\left(T_c^4(t) - T^4(t)\right)\right), \tag{1}$$

$$T(0) = T_0, \quad T(t_f) = T_f, \tag{2}$$

$$\int_0^{t_f} \frac{\varkappa}{T(t)} \exp\left[-\beta/T(t)\right] dt \to \min_{T_c(t)}. \tag{3}$$

The optimal-control problem consists of the choice of conditions of temperature variation of the medium over time such as to ensure ingot heating from a specified initial temperature T_0 to a specified final temperature T_f, with minimum skin formation at time t_f. The control function $T_c(t)$ satisfies the constraint $T_0 < A_1 \le T_c(t) \le T_f \le A_2$ for any $0 \le t \le t_f$ and is a piecewise-continuous function of t with a finite number of points of discontinuity.

Using the mathematical apparatus of [1, 2], the optimal heating trajectory of the metal, the optimal control, and the switching time of the controlling action (temperature of the heating medium) are obtained. The procedure for elucidating the structure of the controlling function, the optimal trajectory, and the algorithm for solving Eqs. (1)-(3) may be found in the Appendix.

After finding the switching time t_2, the optimal control is written in the form

$$T_c(t) = \begin{cases} A_1, & 0 \leqslant t < t_2, \\ A_2, & t_2 \leqslant t \leqslant t_f. \end{cases}$$

Thus, a two-stage heating graph of fine-grade ingots is obtained, as well as an algorithm for determining the switching time.

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